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Diffusive Limit of a Kinetic Model for Cometary Flows

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Abstract A kinetic equation with a relaxation time model for wave-particle collisions is considered. Similarly to the BGK-model of gas dynamics, it involves a projection onto the set of equilibrium distributions, nonlinearly dependent on the moments of the distribution function. Under a diffusive and low Mach number scaling the macroscopic limit is a generalization of the incompressible Navier-Stokes equations, where the momentum equations are coupled to a diffusive equation for an energy distribution function. By a moment approximation, this system can be related to a low Mach number model of fluid mechanics, which already appeared in the literature. Finally, for a linearized version corresponding to Stokes flow an existence result for initial value problems is proved.

Keywords Kinetic equation · Wave-particle collision operator · Cometary flows · Diffusive scaling · Macroscopic limit · Low Mach number model

1 Introduction and basic facts

We consider a kinetic initial value problem in the dimensionless form

$$\frac{\partial f}{\partial t} + \mathrm{v} \cdot \nabla_x f = Q(f), \tag{1}$$

$$f(x, \mathrm{v}, t = 0) = f_I(x, \mathrm{v}), \tag{2}$$

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where $f = f(x, \mathrm{v}, t) \geq 0$ denotes a particle distribution function with the velocity variable $\mathrm{v} \in \mathbb{R}^d$ ($d \geq 1$ being the space dimension) and periodic in the position variable $x \in \mathbb{T}^d$, where $\mathbb{T}^d$ is a d-dimensional torus. The collision operator Q is derived in the quasilinear plasma theory as a simplified model describing wave-particle interaction in cometary flows:

$$Q(f) = P_{u_f}(f) - f, \tag{3}$$

where P_u is a projection on the set of distribution functions isotropic around the velocity $u \in \mathbb{R}^d$:

$$[P_u(f)](\mathrm{v}) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(u + |\mathrm{v} - u|\omega)\,\mathrm{d}\omega, \tag{4}$$

with $|S^{d-1}|$ denoting the $(d-1)$-dimensional Lebesgue measure of the unit sphere S^{d-1} in $\mathbb{R}^d$. The mass, momentum and energy densities associated with f are given by

$$\varrho_f = \int_{\mathbb{R}^d} f\,\mathrm{dv}, \qquad m_f = \int_{\mathbb{R}^d} f\mathrm{v}\,\mathrm{dv}, \qquad E_f = \int_{\mathbb{R}^d} f\frac{|\mathrm{v}|^2}{2}\,\mathrm{dv}. \tag{5}$$

Finally, the mean velocity is defined as the ratio of the macroscopic momentum density and the mass density:

$$u_f = \frac{m_f}{\varrho_f} = \frac{\int_{\mathbb{R}^d} f\mathrm{v}\,\mathrm{dv}}{\int_{\mathbb{R}^d} f\,\mathrm{dv}}. \tag{6}$$

Note that the nonlinearity of Q is induced by the appearance of the mean velocity u_f in the projection (3). The equations (1)–(6) are in dimensionless form; in particular, a relaxation time appearing in the dimensional form of the collision operator has been used as the reference time.

For the physical background of the model we refer to [10, 19, 20] and [21]; let us only remark that the collision operator (3) describes the scattering of cosmic rays (energetic particles) in an astrophysical plasma, caused by random irregularities in the ambient field [10]; the collision operator considered here is a simplified relaxation time model, comparable to the BGK model of gas dynamics. A mathematical treatment of this model has been started in [5]. In [7], the whole space problem is considered and existence of weak solutions of the initial value problem is proven under a positivity assumption for the initial data, guaranteeing that vacuum is avoided. Also, macroscopic conservation laws, an entropy dissipation equality and the propagation of higher order moments are rigorously proven here. In [14], an existence theorem under milder conditions is presented. In particular, the occurrence of vacuum is allowed. The large time behaviour and equilibrium solutions have been studied in [13–15]. In [6], formal macroscopic limits of the model are investigated, both in the hydrodynamic and diffusive scalings. The diffusive case is considered in the framework of small perturbations of global equilibria and the corresponding macroscopic dynamics consists of the incompressible Navier-Stokes equations for the mean velocity coupled with a diffusion equation for the energy distribution function, constrained by a generalized Boussinesq relation. In [18], two macroscopic limits in the diffusive scaling are carried out rigorously. Both of them deal with perturbations of a global Maxwellian with vanishing mean velocity, meaning that the Mach number is small. Depending on the velocity scaling, the macroscopic limit is either the incompressible (actually constant density) Navier-Stokes system or Stokes flow. Here we are also interested in a small Mach number scaling, however, allowing for a position dependence of the dominating distribution function. In the series of works [3, 4] and [8], the model with the collision time depending on the particle energy in the fluid frame is considered in context of turbulence modelling. In [3], a Chapman-Enskog expansion in the hydrodynamic scaling is provided, leading to a generalized Navier-Stokes system. A turbulence model of k-ε type is derived in [4] and compared to the standard k-ε models for incompressible flows, and a macroscopic system of equations including the mean velocity, the turbulent energy and an arbitrary high order velocity moment of the probability distribution function is derived by means of the entropy minimisation principle. Finally, in [8], a new extension of the model (1)–(4) is introduced, in which the collision operator relaxes to anisotropic equilibria. The diffusive corrections of this fluid-dynamical model are investigated using the Chapman-Enskog techniques and the influence of the anisotropic character on the expression of the viscosity and of the heat flux is shown.

After a short presentation of relevant properties of the kinetic equation in Sect. 2, we formally compute the macroscopic limit under a diffusive macroscopic scaling, where the Knudsen number and the Mach number are of the same order of magnitude (Sect. 3). The result is a diffusive equation for an energy distribution function with the side condition of constant pressure, coupled to a Navier-Stokes system for the mean velocity with the special feature that the divergence of the velocity does not vanish, but is given in terms of the energy distribution function. Constant density Navier-Stokes is recovered as a special solution with position independent energy distribution.

In Sect. 4, some properties of the limiting system are discussed, including an entropy inequality. Also a strongly simplified version is obtained by a moment procedure assuming a local Maxwellian energy distribution. This simplification is equivalent to a low Mach number model, which has appeared earlier in the fluid mechanics literature [11, 17]. Only a rather weak existence theory is known for this problem [16], and we were not able to extend it to the system with general energy distribution. Moreover, we show that our limiting system can be obtained as a low Mach number limit of the generalized compressible Navier-Stokes system derived in [3].

A linearization around a global equilibrium is performed in Sect. 5, which results in a Stokes system for the divergence-free part of the velocity and a decoupled equation for the energy distribution, which is nonlocal in terms of the energy. Using an entropy inequality which, surprisingly, seems unrelated to that for the full nonlinear problem, an existence analysis is carried out. A noteworthy property of the problem is that, although being diffusive in the position direction, it does not regularize all solution components. The problem is equivalent, though somewhat differently formulated, to what has been derived as part of the Stokes limit in [18].

2 Preliminaries

We start with collecting some formal properties of the linear collision operator $Q_u(f) = P_u(f) - f$ with a fixed vector $u \in \mathbb{R}^d$; see, e.g., [5, 7].

Lemma 1 *Let* $u \in \mathbb{R}^d$, $f, g \in C_0^\infty(\mathbb{R}^d)$, $f, g \geq 0$, $\varrho_f > 0$, $\psi \in C^\infty([0, +\infty))$. *Then*

(i) *Symmetry*: $\int_{\mathbb{R}^d} Q_u(f) g \, \mathrm{dv} = \int_{\mathbb{R}^d} f Q_u(g) \, \mathrm{dv}$.
(ii) *Collision invariants*: $\int_{\mathbb{R}^d} Q_u(f) \psi(|\mathrm{v} - u|) \, \mathrm{dv} = 0$.
(iii) *Balance of average momentum*: $\int_{\mathbb{R}^d} Q_u(f) \mathrm{v} \, \mathrm{dv} = \varrho_f (u - u_f)$.
(iv) *Equilibrium*: $Q_u(f) = 0$ *if and only if there exists* $F \in C_0^\infty([0, +\infty))$, *such that* $f(\mathrm{v}) = F(|\mathrm{v} - u|^2/2)$.

(v) *H-theorem*: *For* χ *convex*,

$$\int_{\mathbb{R}^d} Q_u(f)\chi'(f)\,\mathrm{dv} = -\int_{\mathbb{R}^d} [f - P_u(f)][\chi'(f) - \chi'(P_u(f))]\,\mathrm{dv} \le 0$$

Remark 1 The statements of Lemma 1 can be extended for less regular functions by density arguments, whenever the involved integrals are well defined. This is the way those results will be used in the following.

From (ii) it follows that the set of collision invariants of $Q = Q_{u_f}$ is infinite dimensional and depends nonlocally on the distribution function f through the mean velocity u_f. This is a distinctive feature of the model compared to the classical Boltzmann theory [2]. From (ii) and (iii) we see that the only f-independent collision invariants of Q are linear combinations of 1, the components of v and $|\mathrm{v}|^2 = |\mathrm{v} - u_f|^2 - |u_f|^2 + 2u_f \cdot \mathrm{v}$. Local conservation laws for mass, momentum and energy are produced by these:

$$\frac{\partial}{\partial t}\begin{pmatrix} \varrho_f \\ m_f \\ E_f \end{pmatrix} + \nabla_x \cdot \int_{\mathbb{R}^d} \begin{pmatrix} \mathrm{v} \\ \mathrm{v}\otimes\mathrm{v} \\ \mathrm{v}|\mathrm{v}|^2/2 \end{pmatrix} f\,\mathrm{dv} = 0. \tag{7}$$

The H-theorem leads to the entropy dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \chi(f)\,\mathrm{dv}\,\mathrm{d}x = -\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} Q(f)\chi'(f)\,\mathrm{dv}\,\mathrm{d}x \le 0 \tag{8}$$

for any convex function χ. The fundamental conservation and dissipation properties (7), (8) have been verified rigorously in [7].

Trivial to prove but important for the forthcoming analysis is the following observation about the solvability of the equation $P_0(f) - f = g$ (0 stands for the zero vector in $\mathbb{R}^d$) for a given function g:

Lemma 2 *The equation* $P_0(f) - f = g$ *is solvable if and only if the condition*

$$\int_{S^{d-1}} g(r\omega)\,\mathrm{d}\omega = 0 \quad \textit{for each } r > 0 \tag{9}$$

is satisfied. Solutions have the form

$$f(\mathrm{v}) = F(|\mathrm{v}|^2/2) - g(\mathrm{v})$$

with an arbitrary function F.

3 Diffusive Scaling and Hilbert Expansion

In this section, we perform the diffusive scaling in (1) and, after a suitable coordinate transformation, Hilbert expansion in terms of the scaling parameter ε. An analysis of the resulting equations allows us then to formally derive a system for the limiting problem when $\varepsilon \to 0$. From now on, we assume for the dimension $d = 3$ and the 3-dimensional torus we denote just by $\mathbb{T}$.

In the diffusive scaling, we assume the macroscopic reference time to be of order ε^{-2} longer than the microscopic reference time, while the macroscopic reference length is of order ε^{-1} larger than the microscopic one, with ε being a small positive parameter. This makes the average velocity of order ε and we introduce the notation $u_f = \varepsilon u$; the scaled version of (1) then reads

$$\varepsilon^2 \frac{\partial f}{\partial t} + \varepsilon \mathrm{v} \cdot \nabla_x f = P_{\varepsilon u}(f) - f \quad \text{with} \int_{\mathbb{R}^3} f(\mathrm{v})\mathrm{v}\, d\mathrm{v} =: \varepsilon \varrho_f u \tag{10}$$

Similarly as in [3], we introduce the coordinate transform $v := \mathrm{v} - \varepsilon u(x,t)$, which leads to

$$\int_{\mathbb{R}^3} f(v) v\, dv = 0 \tag{11}$$

and (10) becomes

$$\begin{aligned} &\varepsilon^2 \frac{\partial f}{\partial t} - \varepsilon^3 \frac{\partial u}{\partial t} \cdot \nabla_v f + \varepsilon v \cdot \nabla_x f + \varepsilon^2 u \cdot \nabla_x f \\ &\quad - \varepsilon^2 v^T \nabla_x u \nabla_v f - \varepsilon^3 u^T \nabla_x u \nabla_v f = P_0(f) - f. \end{aligned} \tag{12}$$

The coordinate transform moves the nonlinearity from the collision operator in (10) to the left hand side of (12), which facilitates the forthcoming analysis. Multiplication of (12) by v and integration yields the momentum conservation equation

$$\varepsilon^2 \varrho_f \left(\frac{\partial u}{\partial t} + (u \cdot \nabla_x) u \right) + \nabla_x \cdot \pi_f = 0, \tag{13}$$

where we denoted the stress tensor by $\pi_f = \int f(v \otimes v)\, dv$. We consider the system (12), (13) for the unknowns f and u, for $x \in \mathbb{T}$, $v \in \mathbb{R}^3$ and $t \ge 0$, subject to the initial conditions

$$\begin{aligned} &f(x,v,0) = f_I(x,v) \quad \text{with} \int_{\mathbb{R}^3} f_I(x,v) v\, dv = 0, \\ &u(x,0) = u_I(x). \end{aligned}$$

The momentum equation then guarantees (11) for all times.

The system (12), (13) is singularly perturbed. In particular, we expect an initial time layer with the time scale $\tau = t/\varepsilon^2$. Introducing τ instead of t and passing to the limit $\varepsilon \to 0$ gives the initial layer equations

$$\frac{\partial f}{\partial \tau} = P_0(f) - f, \tag{14}$$

$$\varrho_f \frac{\partial u}{\partial \tau} + \nabla_x \cdot \pi_f = 0, \tag{15}$$

subject to the initial conditions $f(\tau = 0) = f_I$ and, respectively, $u(\tau = 0) = u_I$. The equation (14) acts only on the v- and τ-variables and is easily solved:

$$f = P_0(f_I) + e^{-\tau}\big(f_I - P_0(f_I)\big).$$

Then we have

$$\nabla_x \cdot \pi_f = \frac{2}{3}(1 - e^{-\tau}) \nabla_x e_{f_I} + e^{-\tau} \nabla_x \cdot \pi_{f_I},$$

where e_{f_I} is the initial thermal energy ($e_f := \int_{\mathbb{R}^3} f|v|^2/2\,dv$, which together with the macroscopic kinetic energy constitutes the total energy, $E_f = e_f + \varepsilon^2 \varrho_f |u_f|^2/2$). Obviously, for (15) to have a bounded solution as $\tau \to \infty$, we need $\nabla_x e_{f_I} = 0$. Under this assumption, observing that $\varrho_f(\tau) \equiv \varrho_{f_I}$ for all $\tau > 0$, we get

$$u = u_I - (1 - e^{-\tau}) \frac{\nabla_x \cdot \pi_{f_I}}{\varrho_I}.$$

For the outer solution of (12), (13), i.e., for the evolution in terms of the slow time variable t, we assume the existence of an asymptotic expansion

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \mathcal{O}(\varepsilon^3), \qquad u = u_0 + \mathcal{O}(\varepsilon), \tag{16}$$

and prescribe the matching condition that the initial data for the outer solution is the limit of the layer solution as $\tau \to \infty$. Accordingly, we prescribe the following initial conditions for f_0 and u_0:

$$f_0(t=0) = P_0(f_I), \qquad u_0(t=0) = u_I - \frac{\nabla_x \cdot \pi_{f_I}}{\varrho_I}.$$

Substitution of the expansion (16) in (12) and comparing terms of order $\mathcal{O}(1)$ implies that $P_0(f) - f = 0$ and, thus, due to Lemma 2,

$$f_0(x,v,t) = F_0(x,\xi,t), \quad \text{with } \xi = |v|^2/2,$$

for an as yet undetermined function F_0. Equating terms of order $\mathcal{O}(\varepsilon)$ in (12) then gives

$$v \cdot \nabla_x F_0 = P_0(f_1) - f_1, \tag{17}$$

and we use Lemma 2 again to obtain

$$f_1 = F_1 - v \cdot \nabla_x F_0 \quad \text{with } F_1 = F_1(x,\xi,t). \tag{18}$$

The solvability condition (9) is obviously satisfied because of the spherical symmetry of F_0 with respect to the velocity.

Equating terms of order $\mathcal{O}(\varepsilon^2)$ in (12) gives:

$$\frac{\partial f_0}{\partial t} + v \cdot \nabla_x f_1 + u_0 \cdot \nabla_x f_0 - v^T \nabla_x u_0 \nabla_v f_0 = P_0(f_2) - f_2. \tag{19}$$

The solvability condition of Lemma 2 for (19) gives an equation for F_0:

$$\frac{\partial F_0}{\partial t} - \frac{2}{3}\xi \Delta_x F_0 + u_0 \cdot \nabla_x F_0 - \frac{2}{3}\xi (\nabla_x \cdot u_0) \frac{\partial F_0}{\partial \xi} = 0, \tag{20}$$

which, after multiplication by $\sqrt{\xi}$, can be written in the conservative form

$$\frac{\partial}{\partial t}(\sqrt{\xi} F_0) + \nabla_x \cdot \left(\sqrt{\xi} F_0 u_0 - \frac{2}{3}\xi^{3/2} \nabla_x F_0 \right) - \frac{\partial}{\partial \xi} \left(\frac{2}{3}\xi^{3/2} (\nabla_x \cdot u_0) F_0 \right) = 0. \tag{21}$$

Defining

$$\pi_0 = \int_{\mathbb{R}^3} F_0 (v \otimes v)\,\mathrm{d}v \quad \text{and} \quad e_0 = \int_{\mathbb{R}^3} F_0 \frac{|v|^2}{2}\,\mathrm{d}v = \int_0^\infty F_0 \xi \, d\mu(\xi),$$

with $d\mu(\xi) := 4\pi\sqrt{2\xi}\,d\xi$, a simple computation gives $\pi_0 = \frac{2}{3}e_0\mathbb{I}$, where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{3\times 3}$. From (13) we deduce that $\nabla_x \cdot \pi_0 = 0$, which implies

$$\nabla_x e_0 = 0. \tag{22}$$

Integration of (21) with respect to ξ yields the continuity equation

$$\frac{\partial \varrho_0}{\partial t} + \nabla_x \cdot (\varrho_0 u_0) = 0, \tag{23}$$

where the mass density ϱ_0 is the limit of ϱ_f as $\varepsilon \to 0$:

$$\varrho_0 = \int_{\mathbb{R}^3} F_0\,dv = \int_0^\infty F_0\,d\mu(\xi).$$

Multiplication of (21) by ξ and integration gives (recalling that $\nabla_x e_0 = 0$)

$$\frac{\partial e_0}{\partial t} - \nabla_x \cdot \left(\frac{2}{3}\nabla_x M_0 - \frac{5}{3}e_0 u_0\right) = 0, \tag{24}$$

with M_0 denoting the fourth-order moment of F_0,

$$M_0 = \int_{\mathbb{R}^3} F_0 \frac{|v|^4}{4}\,dv = \int_0^\infty F_0 \xi^2\,d\mu(\xi).$$

Obviously, integration of (24) with respect to x gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} e_0\,dx = 0, \tag{25}$$

which, in combination with the previous result (22), implies that e_0 is a global constant. Using this information in (24), we find a formula for the divergence of u_0, namely

$$\nabla_x \cdot u_0 = \frac{2}{5e_0}\Delta_x M_0. \tag{26}$$

This leads to an additional compatibility requirement for the initial data:

$$\nabla_x \cdot u_I - \nabla_x \cdot \left(\frac{\nabla_x \cdot \pi_{f_I}}{\varrho_{f_I}}\right) = \frac{2}{5e_{f_I}}\Delta_x M_{f_I}.$$

Let us note that our computations are also valid in the case of a bounded position domain Ω with reflecting boundary conditions,

$$f(x,t,\mathrm{v}) = f(x,t,\mathrm{v}')$$

for $t > 0$, $x \in \partial\Omega$, with specular or inverse reflection, i.e.,

$$\text{a) } \mathrm{v}' = \mathrm{v} - 2(n(x)\cdot\mathrm{v})n(x) \quad \text{or} \quad \text{b) } \mathrm{v}' = -\mathrm{v},$$

where $n(x)$ denotes the unit normal along $\partial\Omega$. Indeed, the global version of the energy conservation (7) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E_f(x,t)\,\mathrm{d}x = \int_{\partial\Omega}\int_{\mathbb{R}^d} f(x,\mathrm{v},t)\frac{|\mathrm{v}|^2}{2}\mathrm{v}\cdot n(x)\,\mathrm{d}\mathrm{v}\,\mathrm{d}x$$

and since in both cases we have $|\mathrm{v}'|^2 = |\mathrm{v}|^2$ and $\mathrm{v}' \cdot n(x) = -\mathrm{v} \cdot n(x)$, the right hand side is zero. Therefore, the equality (25) remains valid and can be combined with (22) to obtain (26).

In a similar way, we could consider the whole space problem $\Omega = \mathbb{R}^3$ if we proposed the initial condition $f_I(x, \mathrm{v}) = F_\infty(|\mathrm{v}|^2/2) + g(x, \mathrm{v})$ with F_∞ a sufficiently fast decaying function and g with compact support in x and zero energy. Then (25) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} (e_0 - e_\infty)\, \mathrm{d}x = 0,$$

with e_∞ denoting the energy of F_∞, and we have $e_0 \equiv e_\infty$.

The Momentum Equation

Equating the $\mathcal{O}(1)$- and $\mathcal{O}(\varepsilon)$-terms in (13) yields $\nabla_x \cdot \pi_0 = 0$, implying the constantness of e_0, and, respectively, $\nabla_x \cdot \pi_1 = 0$, which similarly implies $\nabla_x e_1 = 0$. However, the latter information is not of interest for our purposes. Collecting terms of the order of ε^2 gives

$$\varrho_0 \left(\frac{\partial u_0}{\partial t} + (u_0 \cdot \nabla_x) u_0 \right) + \nabla_x \cdot \pi_2 = 0 \tag{27}$$

with $\pi_2 = \int f_2 (v \otimes v)\, \mathrm{d}v$.

Solving equation (19) for f_2 yields

$$f_2 = F_2 - \frac{\partial F_0}{\partial t} - v \cdot \nabla_x f_1 - u_0 \cdot \nabla_x F_0 + v_i \frac{\partial u_{0j}}{\partial x_i} \frac{\partial F_0}{\partial \xi} v_j,$$

where $F_2 = F_2(x, \xi, t)$ and the summation convention has been used in the last term. After substitution for f_1 from (18) and $\frac{\partial F_0}{\partial t}$ from (20), we get

$$\begin{aligned} f_2 = F_2 - v \cdot \nabla_x F_1 + \nabla_x \cdot \left[\left(v \otimes v - \frac{2\xi}{3} \mathbb{I} \right) \nabla_x F_0 \right] \\ + \left(v^T \nabla_x u_0 v - \frac{2\xi}{3} \nabla_x \cdot u_0 \right) \frac{\partial F_0}{\partial \xi}. \end{aligned}$$

From this expression we shall calculate $\pi_2 = \int f_2 (v \otimes v)\, \mathrm{d}v$, which involves not very difficult, but tedious calculations. We start with the following auxiliary results:

$$\int_{S^2} \omega_i^2 \omega_j^2 \, \mathrm{d}\omega = \frac{4\pi}{15}, \quad i \neq j; \qquad \int_{S^2} \omega_i^4 \, \mathrm{d}\omega = \frac{4\pi}{5}; \qquad \int_{S^2} \omega_i^2 \, \mathrm{d}\omega = \frac{4\pi}{3}.$$

Another useful observation is that whenever we integrate (with respect to v over $\mathbb{R}^3$) an expression involving a spherically symmetric function (F_0, F_1 or F_2 in particular) multiplied by an odd product of the components of v, we obviously obtain zero as the result.

The computation is carried out in four parts, corresponding to the four terms in the expression for f_2.

(i) Using the notation $p_2 := \int_0^\infty F_2(\xi)(2\xi)^2 \sqrt{2\xi}\, \mathrm{d}\xi$, we have

$$\int_{\mathbb{R}^3} F_2 (v \otimes v)\, \mathrm{d}v = p_2 \mathbb{I}.$$

(ii) Due to the spherical symmetry of F_1, we have

$$\int_{\mathbb{R}^3} v \cdot \nabla_x F_1 (v \otimes v)\, \mathrm{d}v = 0.$$

(iii) We denote

$$\left[\int_{\mathbb{R}^3} (v \otimes v) \nabla_x \cdot \left(v \otimes v - \frac{2\xi}{3} \mathbb{I} \right) \nabla_x F_0 \, \mathrm{d}v \right]_{ij}$$
$$= \sum_{k,l=1}^{3} \int_{\mathbb{R}^3} v_i v_j \left(v_k v_l - \frac{2\xi}{3} \delta_{kl} \right) \frac{\partial^2 F_0}{\partial x_k \partial x_l} \, \mathrm{d}v =: A_{ij},$$

and with this notation, we calculate for $i = j$:

$$\begin{aligned} A_{ii} &= \sum_{k=1}^{3} \int_{\mathbb{R}^3} v_i^2 \left(v_k^2 - \frac{2\xi}{3} \right) \frac{\partial^2 F_0}{\partial x_k^2} \, \mathrm{d}v \\ &= \sum_{k=1}^{3} \int_{S^2} \omega_i^2 \left(\omega_k^2 - \frac{1}{3} \right) \mathrm{d}\omega \frac{1}{\pi} \int_0^\infty \xi^2 \frac{\partial^2 F_0}{\partial x_k^2} d\mu(\xi) \\ &= \sum_{k=1}^{3} \left(\int_{S^2} \omega_i^2 \omega_k^2 \, \mathrm{d}\omega - \frac{1}{3} \int_{S^2} \omega_i^2 \, \mathrm{d}\omega \right) \frac{1}{\pi} \frac{\partial^2 M_0}{\partial x_k^2} \\ &= \sum_{k=1}^{3} \left(\frac{8}{15} \delta_{ik} - \frac{8}{45} \right) \frac{\partial^2 M_0}{\partial x_k^2} = \frac{8}{15} \left(\frac{\partial^2 M_0}{\partial x_i^2} - \frac{1}{3} \Delta_x M_0 \right). \end{aligned}$$

For $i \neq j$ we then have

$$\begin{aligned} A_{ij} &= 2 \int_{\mathbb{R}^3} v_i^2 v_j^2 \frac{\partial^2 F_0}{\partial x_i \partial x_j} \, \mathrm{d}v = \frac{2}{\pi} \int_{S^2} \omega_i^2 \omega_j^2 \, \mathrm{d}\omega \frac{\partial^2}{\partial x_i \partial x_j} \int_0^\infty \xi^2 F_0 d\mu(\xi) \\ &= \frac{8}{15} \frac{\partial^2 M_0}{\partial x_i \partial x_j}. \end{aligned}$$

(iv) Again, we denote

$$\left[\int_{\mathbb{R}^3} (v \otimes v) \left(v^T \nabla_x u_0 v - \frac{2\xi}{3} \nabla_x \cdot u_0 \right) \frac{\partial F_0}{\partial \xi} \, \mathrm{d}v \right]_{ij} =: B_{ij},$$

and in the very similar way as in the preceding case we obtain the following results:

$$\begin{aligned} B_{ii} &= \frac{4}{3} \left[\frac{1}{3} \nabla_x \cdot u_0 - \frac{\partial u_{0i}}{\partial x_i} \right] e_0, \\ B_{ij} &= -\frac{2}{3} \left(\frac{\partial u_{0i}}{\partial x_j} + \frac{\partial u_{0j}}{\partial x_i} \right) e_0 \quad \text{for } i \neq j. \end{aligned}$$

Altogether, we have

$$\pi_2 = p_2 \mathbb{I} + \frac{8}{15} \left(H(M_0) - \frac{1}{3} (\Delta_x M_0) \mathbb{I} \right) + \frac{2}{3} \left(\frac{2}{3} (\nabla_x \cdot u_0) \mathbb{I} - \sigma(u_0) \right) e_0,$$

where we denoted by $H(M_0)$ the Hessian matrix of M_0 and $\sigma(u_0)_{ij} = \frac{\partial u_{0i}}{\partial x_j} + \frac{\partial u_{0j}}{\partial x_i}$. Now, using the identities

$$\nabla_x \cdot H(M_0) = \nabla_x(\Delta_x M_0), \qquad \nabla_x \cdot \sigma(u_0) = \Delta_x u_0 + \nabla_x(\nabla_x \cdot u_0),$$

and keeping in mind that $\nabla_x e_0 = 0$, we can finally write

$$\begin{aligned}\nabla_x \cdot \pi_2 &= \nabla_x \left(p_2 + \frac{16}{45}\Delta_x M_0 - \frac{2e_0}{9}\nabla_x \cdot u_0 \right) - \frac{2e_0}{3}\Delta_x u_0 \\ &= \nabla_x p - \frac{2e_0}{3}\Delta_x u_0,\end{aligned}$$

with the obvious definition of p. Substituting this expression in the momentum equation (27) yields

$$\varrho_0 \left(\frac{\partial u_0}{\partial t} + (u_0 \cdot \nabla_x) u_0 \right) + \nabla_x p = \frac{2e_0}{3}\Delta_x u_0. \tag{28}$$

4 Some Properties of the Limiting System

In the previous section, we formally obtained $f(x,v,t) \to F_0(x, |v|^2/2, t)$ and $u(x,t) \to u_0(x,t)$. For these quantities we derived the system

$$\frac{\partial F_0}{\partial t} - \frac{2}{3}\xi \Delta_x F_0 + u_0 \cdot \nabla_x F_0 - \frac{2}{3}\xi(\nabla_x \cdot u_0)\frac{\partial F_0}{\partial \xi} = 0 \tag{29}$$

$$\nabla_x \cdot u_0 = \frac{2}{5e_0}\Delta_x M_0 \tag{30}$$

$$\varrho_0 \left(\frac{\partial u_0}{\partial t} + u_0 \cdot \nabla_x u_0 \right) + \nabla_x p = \frac{2e_0}{3}\Delta_x u_0, \tag{31}$$

which we consider for $x \in \mathbb{T}$, $\xi > 0$ and $t > 0$, subject to the initial conditions

$$F_0(x,\xi,0) = F_{0,I}(x,\xi), \qquad u_0(x,0) = u_{0,I}(x),$$

satisfying

$$\nabla_x e_{F_{0,I}} = 0, \qquad \nabla_x \cdot u_{0,I} = \frac{2}{5e_{F_{0,I}}}\Delta_x M_{F_{0,I}}.$$

Let us recall that the mass, energy, and fourth-order moment densities related to F_0 are defined as

$$\varrho_0 = \int_0^\infty F_0 \, d\mu(\xi), \qquad e_0 = \int_0^\infty F_0 \xi \, d\mu(\xi), \qquad M_0 = \int_0^\infty F_0 \xi^2 \, d\mu(\xi).$$

The system (29)–(31) is a generalization of the incompressible Navier-Stokes equations for u_0. With $F_{0,I} = F_{0,I}(\xi)$, a solution is given by $F_0 \equiv F_{0,I}$ and u_0 a solution of the incompressible Navier-Stokes equations with constant viscosity $2e_0/3$ and constant density ϱ_0. In the full system, the velocity is not divergence free, but the pressure p in (31) can still be seen as Lagrange multiplier for the constraint (30).

Analogously to (8), an entropy inequality holds for F_0. For χ convex, we multiply equation (29) with $\chi'(F_0)$ and integrate:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}\int_0^\infty \chi(F_0)d\mu(\xi)dx+\frac{2}{3}\int_{\mathbb{T}}\int_0^\infty \xi|\nabla_x F_0|^2\chi''(F_0)d\mu(\xi)dx\\&\quad-\int_{\mathbb{T}}\int_0^\infty u_0\cdot\nabla_x\chi(F_0)d\mu(\xi)dx\\&\quad+\frac{2}{3}\int_{\mathbb{T}}\int_0^\infty(\nabla_x\cdot u_0)\left(\frac{\partial}{\partial\xi}\chi(F_0)\right)\xi\, d\mu(\xi)dx=0.\end{aligned}$$

Integrations by parts with respect to x and ξ in the third and fourth term, respectively, yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}\int_0^\infty \chi(F_0)d\mu(\xi)dx=-\frac{2}{3}\int_{\mathbb{T}}\int_0^\infty \xi|\nabla_x F_0|^2\chi''(F_0)d\mu(\xi)dx\le 0. \tag{32}$$

This a priori estimate should open the way to prove existence of solutions. However, our attempts failed due to the inability to control the moment M_0 (notice that the system of moments induced by (29) is not closed). A simple calculation provides a plausible excuse for this failure: namely, multiplication of (29) by ξ^2 and integration with respect to ξ and x, using (30), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} M_0\,\mathrm{d}x=\frac{8}{15e_0}\int_{\mathbb{T}}|\nabla_x M_0|^2\,\mathrm{d}x, \tag{33}$$

suggesting ill-posedness.

The following consideration gives another indication of the complexity of the problem: We approximate the solution of (29) by a moment method, where a moment closure is achieved by the Maxwellian ansatz

$$F_0(x,\xi,t)=\frac{\varrho_0(x,t)}{(2\pi T_0(x,t))^{3/2}}e^{-\xi/T_0(x,t)}, \tag{34}$$

where ϱ_0 is the mass density associated with F_0 and $T_0(x,t)$ the temperature. Then, for the energy density e_0 we have $e_0=\frac{3}{2}\varrho_0 T_0$ and, denoting the constant $2e_0/3$ by μ_0, we obtain the relation $\varrho_0 T_0=\mu_0$. Using this result in the formula for M_0, we have

$$M_0=\frac{15}{4}\varrho_0 T_0^2=\frac{15\mu_0^2}{4\varrho_0}.$$

Therefore, with (23), (28), and (30), we have the closed system of equations

$$\frac{\partial\varrho_0}{\partial t}+\nabla_x\cdot(\varrho_0 u_0)=0, \tag{35}$$

$$\nabla_x\cdot u_0=\mu_0\Delta_x(1/\varrho_0), \tag{36}$$

$$\varrho_0\left(\frac{\partial u_0}{\partial t}+(u_0\cdot\nabla_x)u_0\right)+\nabla_x p=\mu_0\Delta_x u_0, \tag{37}$$

which is equivalent to the low Mach number model with constant viscosity, introduced (as a part of a model for reactive flows) in [17] and studied in particular in [11], as far as the local-in-time well-posedness is concerned. The system can be derived from the gas dynamics

Boltzmann equation or from the BGK-model by scaling assumptions analogous to ours, or as the low Mach number limit of the compressible Navier-Stokes system, as presented in [16]. Here also the best available result on (35)–(37) is given, which consists of local in time existence in a spatially two-dimensional setting with small initial data. These rather restrictive assumptions are an indication that any generalization would be hard to achieve, in particular to the much more general system (29)–(31). Therefore, in the following Section we shall restrict our attention to its linearized version only.

Finally, we show that the system (29)–(31) can be obtained as the low Mach number limit of the system (58) of [3], which with our notation reads

$$\begin{aligned}
&\frac{\partial F}{\partial t} + u \cdot \nabla_x F - (\nabla_x \cdot u)\frac{2\xi}{3}\frac{\partial F}{\partial \xi} \\
&\quad = \frac{2\varepsilon}{3}\xi \left(\nabla_x - C_u \frac{\partial}{\partial \xi}\right) \cdot \left(\nabla_x - C_u \frac{\partial}{\partial \xi}\right) F \\
&\qquad + \varepsilon \frac{4}{15\sqrt{\xi}} \frac{\partial}{\partial \xi}\left(\xi^{5/2} \frac{\partial g}{\partial \xi}\right)(\sigma(u) : \nabla_x u), \\
&\varrho \left(\frac{\partial u}{\partial t} + u \cdot \nabla_x u\right) + \frac{2}{3}\nabla_x e = \varepsilon \frac{2e}{3}\Delta_x u,
\end{aligned}$$

with

$$C_u = \frac{\partial u}{\partial t} + u \cdot \nabla_x u, \qquad \sigma(u) = \nabla_x u + (\nabla_x u)^T - \frac{2}{3}(\nabla_x \cdot u) I,$$

and $\varepsilon > 0$ being a (small) parameter, measuring the strength of the diffusive correction. The system has been obtained following a Chapman-Enskog expansion of the kinetic model (1) in hydrodynamic scaling and can be seen as a generalization of the compressible Navier-Stokes equations. Assuming that the velocity u is small of order ε, $u = \varepsilon u_0$, performing a scaling of the microscopic time unit $t \mapsto \varepsilon t$ and writing $F = F_0 + \mathcal{O}(\varepsilon)$, one can divide the first equation by ε and let $\varepsilon \to 0$, which yields (29) for F_0. From the second equation one gets $\nabla_x e = \mathcal{O}(\varepsilon^2)$, and writing $e = e_0 + \mathcal{O}(\varepsilon)$ gives $\nabla_x e_0 = 0$. Then, the second equation, after division by ε^2 and proper definition of the pressure, gives (31) for u_0 and e_0. As shown in Sect. 3, the constraint $\nabla_x e_0 = 0$ is equivalent to (30).

5 The Stokes Limit

The system (29)–(31) possesses a large family of steady states with constant velocity u_∞ and x-independent, but otherwise arbitrary, energy distribution $F_\infty(\xi)$. For a nonmoving ($u_\infty = 0$) steady state we introduce the perturbations $F(x, \xi, t) = F_0(x, \xi, t) - F_\infty(\xi)$, $u(x, t) = u_0(x, t)$ and assuming their smallness and neglecting quadratic terms in (29)–(31), we obtain the linear system

$$\frac{\partial F}{\partial t} - \frac{2}{3}\xi \Delta_x F - \frac{2}{3}\xi (\nabla_x \cdot u) F'_\infty = 0 \tag{38}$$

$$\nabla_x \cdot u = \frac{2}{5e_\infty}\Delta_x M_F \tag{39}$$

$$\varrho_\infty \frac{\partial u}{\partial t} + \nabla_x p = \frac{2e_\infty}{3}\Delta_x u, \tag{40}$$

where we denoted M_F the fourth order moment of F, $\varrho_\infty := \varrho_{F_\infty}$ and $e_\infty := e_{F_\infty}$. For proving the well-posedness of this system we shall need some assumptions on the smoothness and qualitative behaviour of F_∞. For the moment $0 < \varrho_\infty < \infty$ and $0 < e_\infty < \infty$ suffice. Equation (39) can be written in the form

$$\nabla_x \cdot w = 0, \quad \text{with } w := u - \frac{2}{5e_\infty}\nabla_x M_F.$$

Writing the linearized momentum equation (40) in terms of the new variable w, we get

$$\frac{2\varrho_\infty}{5e_\infty}\nabla_x\left(\frac{\partial M_F}{\partial t}\right) + \varrho_\infty\frac{\partial w}{\partial t} + \nabla_x p = \frac{4}{15}\nabla_x(\Delta_x M_F) + \frac{2e_\infty}{3}\Delta_x w,$$

and, after a redefinition of the pressure, the initial value problem for (38)–(40) breaks into two decoupled parts, namely, Stokes flow for w,

$$\varrho_\infty\frac{\partial w}{\partial t} + \nabla_x p = \frac{2e_\infty}{3}\Delta_x w, \qquad \nabla_x \cdot w = 0, \tag{41}$$

subject to an initial condition of the form

$$w(x,0) = w_I(x) \quad \text{with } \nabla_x \cdot w_I = 0, \tag{42}$$

and an equation for F,

$$\frac{\partial F}{\partial t} = \frac{2}{3}\xi\Delta_x F + \frac{4}{15e_\infty}\xi(\Delta_x M_F)F'_\infty, \tag{43}$$

subject to the initial condition

$$F(x,\xi,0) = F_I(x,\xi). \tag{44}$$

The Stokes system (41), (42) is a well studied problem. For its analysis we refer for example to [12]. It is important to remember, however, that the solution of the Stokes problem is not the full fluid velocity.

The equation (43) is a linear but nonlocal (in ξ, due to the occurrence of M) equation for F. As a consequence of the asymptotics of the previous section, we should look for solutions satisfying $\nabla_x e_F = 0$, which, at least formally, exist as a consequence of $\partial e_F/\partial t = 0$ under the assumption $\nabla_x e_{F_I} = 0$. For our following investigations, the latter assumption is not necessary, however.

Equilibrium

Writing (43) in the form

$$\frac{\partial F}{\partial t} = \frac{2}{3}\xi\Delta_x\left(F + \frac{2}{5e_\infty}M_F F'_\infty\right),$$

it is plausible that for a steady state $\overline{F}(x,\xi)$ the term under the Laplacian is independent of x, implying the form

$$\overline{F}(x,\xi) = \tilde{F}(\xi) + a(x)F'_\infty(\xi). \tag{45}$$

The problem of identifying $\tilde{F}$ and a corresponding to the initial datum F_I can be resolved by using the conservation laws

$$\frac{\partial}{\partial t}\int_{\mathbb{T}} F(t,x,\xi)dx = 0, \qquad \frac{\partial}{\partial t} e_F(t,x) = 0.$$

In spite of an ambiguity in the choices of $\tilde{F}$ and a (a is determined up to an additive constant), the equilibrium solution is uniquely given by

$$\overline{F}(x,\xi) = \frac{1}{\text{meas}(\mathbb{T})}\int_{\mathbb{T}} F_I(y,\xi)dy - \frac{2}{3\varrho_\infty}\left[e_I(x) - \frac{1}{\text{meas}(\mathbb{T})}\int_{\mathbb{T}} e_I(y)dy\right] F'_\infty(\xi).$$

Entropy—Decay to Equilibrium

As a candidate for an entropy we look for a weigthed L^2-norm such that both terms on the right hand side of (43) are symmetric with respect to the corresponding scalar product. For the first term the weight has to be independent from x, and considering also the second term, an appropriate choice is the weight $\xi/|F'_\infty(\xi)|$. Obviously this requires additional assumptions on F_∞. We allow for two different cases: either $F'_\infty(\xi) < 0$ for all $\xi \geq 0$, or F_∞ has compact support $[0,\xi_0]$ with $F'_\infty(\xi) < 0$ for $0 \leq \xi < \xi_0$. In the second case we consider only solutions of (43) with $F(t,x,\xi) = 0$ for $\xi \geq \xi_0$. Note that this property is conserved by (43) if it holds for the initial data. The monotonicity of the equilibrium distribution already occurred as a stability condition in related situations [9, 15].

Taking the scalar product of (43) with F gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}\int_0^\infty F^2 \frac{\xi}{|F'_\infty|} d\mu(\xi)dx &= -\frac{2}{3}\int_{\mathbb{T}}\int_0^\infty |\nabla_x F|^2 \frac{\xi^2}{|F'_\infty|} d\mu(\xi)dx \\ &\quad + \frac{4}{15 e_\infty}\int_{\mathbb{T}} |\nabla_x M_F|^2 dx. \qquad (46)\end{aligned}$$

An application of the Cauchy-Schwartz inequality leads to the estimate

$$\begin{aligned}|\nabla_x M_F|^2 &\leq \left(\int_0^\infty \xi^2 |\nabla_x F| d\mu(\xi)\right)^2 \leq \int_0^\infty |\nabla_x F|^2 \frac{\xi^2}{|F'_\infty|} d\mu(\xi) \int_0^\infty \xi^2 |F'_\infty| d\mu(\xi) \\ &= \frac{5 e_\infty}{2}\int_0^\infty |\nabla_x F|^2 \frac{\xi^2}{|F'_\infty|} d\mu(\xi),\end{aligned}$$

showing that the right hand side of (46) is nonpositive:

$$\frac{d}{dt}\int_{\mathbb{T}}\int_0^\infty F^2 \frac{\xi}{|F'_\infty|} d\mu(\xi)dx \leq 0. \qquad (47)$$

The estimate also shows that equality in (47) can only be achieved when (equality in the Cauchy-Schwartz inequality) $|\nabla_x F(x,\xi)| = h(x)|F'_\infty(\xi)|$ with an appropriate $h(x)$. This implies that F has the equilibrium form (45). So the decay of the entropy only stops when equilibrium is reached.

Existence

In this section we provide an existence proof for solutions of (43), based on the entropy inequality (47). Although the equation is linear, the existence theory is nontrivial due to the following facts:

(i) the nonlocality in ξ caused by the occurrence of the fourth-order moment M_F;
(ii) the fact that the equation is non-uniformly parabolic in x, due to the factor ξ in front of $\Delta_x F$;
(iii) the fact that the equation actually cannot be parabolic at all, since the energy density $e_F(x)$ remains unchanged (and, thus, in particular unsmoothed).

On the other hand, these problems can be overcome rather easily due to the a priori estimate, which is a consequence of the entropy inequality.

We start by considering the regularized equation

$$\frac{\partial F_R}{\partial t} = \left(\frac{2}{3}\xi_R + \frac{1}{R}\right)\Delta_x F_R + \frac{4\xi_R}{15 e_{R,\infty}} F'_\infty \Delta_x M_R[F_R], \tag{48}$$

with $R > 0$ and

$$\xi_R = \min\{\xi, R\}, \qquad e_{R,\infty} = \int_0^R \xi F_\infty d\mu(\xi), \qquad M_R[F] = \int_0^\infty \xi_R^2 F\, d\mu(\xi).$$

For $R \to \infty$ the regularized equation formally tends to (43). The regularization has been chosen such that for $0 < R < \infty$ the operator on the right hand side is uniformly elliptic with respect to x, and the entropy estimate carries over in the stronger form

$$\frac{d}{dt}\int_{\mathbb{T}}\int_0^\infty F_R^2 \frac{\xi_R}{|F'_\infty|} d\mu(\xi) dx \le -\frac{1}{R}\int_{\mathbb{T}}\int_0^\infty |\nabla_x F_R|^2 \frac{\xi_R}{|F'_\infty|} d\mu(\xi) dx. \tag{49}$$

This already suggests a solution theory for the regularized equation in the space $L^2(\xi_R/|F'_\infty| d\mu(\xi) dx)$. The only missing ingredient is to show that $M_R[F_R]$ is well defined for F_R in that space, but this follows from the estimate

$$\begin{aligned} M_R[F_R]^2 &\le \int_0^\infty \xi_R^3 |F'_\infty| d\mu(\xi) \int_0^\infty F_R^2 \frac{\xi_R}{|F'_\infty|} d\mu(\xi) \\ &= \frac{7}{2}\int_0^R \xi^2 F_\infty d\mu(\xi) \int_0^\infty F_R^2 \frac{\xi_R}{|F'_\infty|} d\mu(\xi). \end{aligned}$$

This suffices for proving the existence of a unique solution of the initial value problem for the regularized equation with initial data $F_I \in L^2(\xi_R/|F'_\infty| d\mu(\xi) dx)$ [1]. The above arguments imply the bounds

$$\begin{aligned} &\int_{\mathbb{T}}\int_0^\infty F_R^2 \frac{\xi_R}{|F'_\infty|} d\mu(\xi) dx \le \int_{\mathbb{T}}\int_0^\infty F_I^2 \frac{\xi_R}{|F'_\infty|} d\mu(\xi) dx, \\ &\int_{\mathbb{T}} M_R[F_R]^2 dx \le \frac{7}{2}\int_0^R \xi^2 F_\infty d\mu(\xi) \int_{\mathbb{T}}\int_0^\infty F_I^2 \frac{\xi_R}{|F'_\infty|} d\mu(\xi)\, \mathrm{d}x. \end{aligned}$$

Under appropriate assumptions on the data these are uniform in R, which is sufficient for passing to the limit $R \to \infty$. The limit of the first estimate provides uniqueness. Finally, we collect our results:

Theorem 1 *Let* $F_\infty = F_\infty(\xi) \in C^1([0,+\infty))$ *be nonnegative with either* $F_\infty > 0$ *and* $F'_\infty < 0$ *on* $[0,\infty)$, *or* $\mathrm{supp} F_\infty = [0,\xi_0]$ *and* $F'_\infty < 0$ *on* $[0,\xi_0)$. *Let* $\int_0^\infty \xi^2 F_\infty d\mu(\xi) < \infty$ *and* $F_I \in L^2(\xi/|F'_\infty| d\mu(\xi)dx)$. *Then there exists a unique global solution* $F \in L^\infty((0,\infty), L^2(\xi/|F'_\infty| d\mu(\xi)dx))$ *of* (43), (44).

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